Journal of Statistical Physics, Vol. 121, Nos. 5/6, December 2005 (© 2005)
DOI: 10.1007/s10955-005-6985-2

Dynamics of Triangulations

P. Collet[1] and J.-P. Eckmann[2,3]

Received December 22, 2004; accepted May 19, 2005

We study a few problems related to Markov processes of flipping triangulations of the sphere. We show that these processes are ergodic and mixing, but find a natural example which does not satisfy detailed balance. In this example, the expected distribution of the degrees of the nodes seems to follow the power law d^{-4}.

KEY WORDS: Ergodicity; detailed balance; power laws.

1. INTRODUCTION

We consider a Markov chain on triangulations of the sphere. Let $\mathcal{T}$ denote the set of triangulations, by this we mean the set of all combinatorially distinct rooted simplicial 3-polytopes.

Tutte[(7)] showed that their number is asymptotically

$$Z_n = \frac{3}{16\sqrt{6\pi n^5}} \left(\frac{256}{27}\right)^{n-2}, \tag{1.1}$$

as the number n of vertices goes to ∞. Of course, Euler's theorem holds for such triangulations, and this means that when there are n nodes, there are also $3n-6$ links and $2n-4$ triangles.

For an element $T \in \mathcal{T}$, we denote by $\mathcal{N}(T)$ the set of nodes and by $\mathcal{L}(T)$ the set of links.

[1]Centre de Physique Théorique, CNRS UMR 7644, Ecole Polytechnique, F-91128 Palaiseau Cedex, France.
[2]Département de Physique Théorique, Université de Genève; e-mail: jean-pierre.eckmann@physics.unige.ch
[3]Section de Mathématiques, Université de Genève.

For any link ℓ (connecting the nodes A and B), we consider the "complementary" link ℓ', which is defined as follows: if (A, B, C) and (A, B, D) are the two triangles sharing the link ℓ, then ℓ' is the link connecting C and D.

We assume that for any $T \in \mathcal{T}$, a probability $\mathbf{P}_T$ is given on $\mathcal{L}(T)$, i.e., $\sum_\ell \mathbf{P}_T(\ell) = 1$. We define a Markov chain on $\mathcal{T}$ as follows. We first choose a link $\ell \in \mathcal{L}(T)$ at random (with probability $\mathbf{P}_T(\ell)$).

- If the link ℓ' belongs to $\mathcal{L}(T)$, we do not change T and proceed with the next independent choice of a link.
- If ℓ' does not belong to $\mathcal{L}(T)$, we erase ℓ and replace it by ℓ'. We obtain in this way a new triangulation T' and we proceed with the next independent choice of a link. This replacement of ℓ by ℓ' is commonly called a *flip* see ref. 6, or a Gross-Varsted move.[5] (see Fig. 1).

We will denote by $\mathbf{P}(T'|T)$ the transition probability of this Markov chain.

2. PROPERTIES OF THE MARKOV CHAIN

We now fix n and let $\mathcal{T}_n$ denote those triangulations with n nodes.

Proposition 2.1. Assume that $\inf_{T \in \mathcal{T}_n} \inf_{\ell \in \mathcal{L}(T)} \mathbf{P}_T(\ell) > 0$. Then the Markov chain defined in Section 1 is irreducible and aperiodic.

Proof. It is well known (see ref. 6) that by flipping links as described above one can connect any two triangulations of $\mathcal{T}_n$ (one shows that any T can be flipped a finite number of times to reach a "Christmas tree" configuration).

Since by our hypothesis any such (finite) succession of moves has a non-zero probability this shows the irreducibility of the chain. To prove aperiodicity, we have to prove that for any high enough iterate of the transition matrix, all the entries are positive. By the previously mentioned

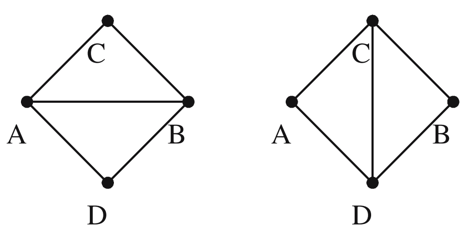

Fig. 1. A flip: the link (A–B) is exchanged with (C–D).

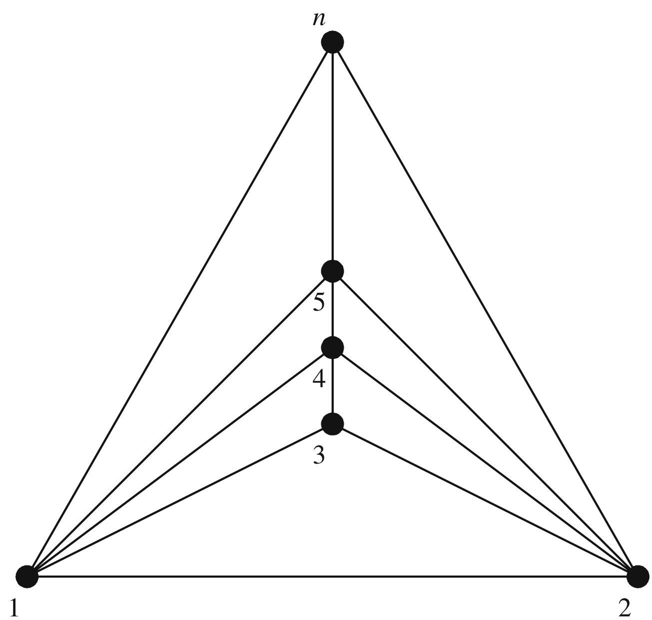

Fig. 2. The "christmas tree" with n nodes, the "branches" between 5 and n not being shown. Any triangulation can be brought to this form by a sequence of flips.

result, it is enough to show that we can construct cycles of length two and three for the "Christmas tree". Cycles of length two are easily obtained by flipping a link back and forth. For cycles of length three, we consider the sub "Christmas tree" of size six at the base of the complete "Christmas tree" (see Fig. 2). We enumerate the nodes as in the figure, assuming $n \geqslant 7$. In particular node 3 has degree 3, nodes 4 to n have degree 4 and nodes 1 and 2 degree $n-2$.

The cycle of length 3 is obtained by performing the following flips

$$\begin{aligned}(1-4) &\to (3-5)\\ (2-3) &\to (1-4)\\ (4-5) &\to (2-3)\end{aligned}$$

after which we get again the "Christmas tree" with nodes 3 and 4 exchanged(Fig. 3). ▮

Remark 2.2. Of course, the condition of Proposition 2.1 is not necessary, but we do not know any simple other criterion in terms of the $\mathbf{P}_T$, but one can think for example of conditions involving two successive flips.

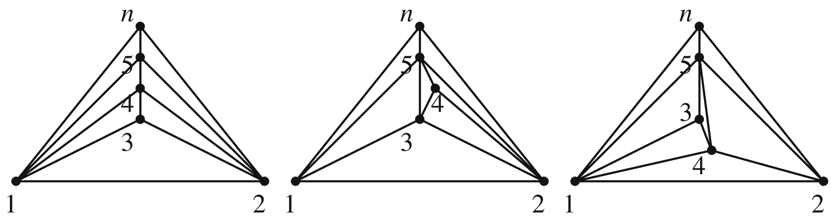

Fig. 3. The three stages in the cycle of 3 flips regenerating the christmas tree with n nodes.

From this result we conclude that there is only one invariant probability measure, and with this measure the chain is ergodic and mixing.

3. TWO EXAMPLES

The easiest example is that where one chooses a link uniformly at random. Then one gets the uniform distribution on $\mathcal{T}$, and, using this simple fact, many properties of this process can be deduced, (see, e.g., ref. 2).

Here, we consider another example, which was suggested to us by Magnasco.[3,4] This process consists in first choosing a node uniformly and then to choose uniformly a link from this node. Let n be the number of nodes. An easy computation, shown below, leads to

$$\mathbf{P}_T(\ell)=\frac{1}{n}\left(\frac{1}{d_1(\ell|T)}+\frac{1}{d_2(\ell|T)}\right) , \tag{3.1}$$

where $d_1(\ell|T)$ and $d_2(\ell|T)$ are the degrees of the nodes at the ends of link ℓ in the triangulation T.

Proof. If ℓ is a link, we denote by $\partial\ell$ the two nodes it connects. If ℓ is a link and i is a node, we say that $\ell\sim i$ if $i\in\partial\ell$. If i is a node, we denote by $d_i(T)$ its degree in the triangulation T. We have from Bayes' formula

$$\mathbf{P}_T(\ell)=\sum_{i\in\mathcal{N}}\mathbf{P}_T(\ell\,|\,i)\mathbf{P}(i).$$

Moreover, $\mathbf{P}(i)=1/n$ for any i, $\mathbf{P}_T(\ell\,|\,i)=0$ if $\ell\not\sim i$ and otherwise

$$\mathbf{P}_T(\ell\,|\,i)=\frac{1}{d_i(T)}.$$

Therefore

$$\mathbf{P}_T(\ell) = \frac{1}{n} \sum_{i \in \partial\ell} \frac{1}{d_i(T)},$$

which is formula (3.1). ∎

It also follows directly from this expression that for any $T \in \mathcal{T}$,

$$\begin{aligned}\sum_{\ell \in T} \mathbf{P}_T(\ell) &= \frac{1}{n} \sum_{\ell \in T} \sum_{i \in \partial\ell} \frac{1}{d_i(T)},\\ &= \frac{1}{n} \sum_{i} \frac{1}{d_i(T)} \sum_{\ell \in \mathcal{L}(T), \ell \sim i} 1 = \frac{1}{n} \sum_{i} \frac{1}{d_i(T)} d_i(T) = 1.\end{aligned}$$

In this computation we have not used the fact that T is a triangulation. Therefore this relation holds for any graph.

For the second model, we have

Theorem 3.1. The Markov chain $\mathbf{P}(\cdot|\cdot)$ is not reversible (when $n \geqslant 7$).

Remark. We have not checked what happens for smaller n.

In other words, one cannot easily guess the invariant measure from the transition probabilities.

Proof. Assume the chain is reversible with respect to some probability $\mathbf{P}$ on $\mathcal{T}$, namely for any T and T' in $\mathcal{T}$ we have

$$\mathbf{P}(T'|T)\mathbf{P}(T) = \mathbf{P}(T|T')\mathbf{P}(T'). \tag{3.2}$$

If $T_1, \ldots, T_k, T_{k+1} = T_1$ is any cycle of admissible flips, we must have

$$\prod_{j=1}^{k} \frac{\mathbf{P}(T_j|T_{j+1})}{\mathbf{P}(T_{j+1}|T_j)} = 1.$$

We are going to show that there is a cycle of length 4 for the christmas graph for which this is not true, (see Fig. 4).

Consider the following cycle for the "Christmas tree" with the same notations as before

$$\begin{aligned}(1-4) &\to (3-5),\\ (2-5) &\to (4-6),\\ (3-4) &\to (2-5),\\ (5-6) &\to (1-4).\end{aligned}$$

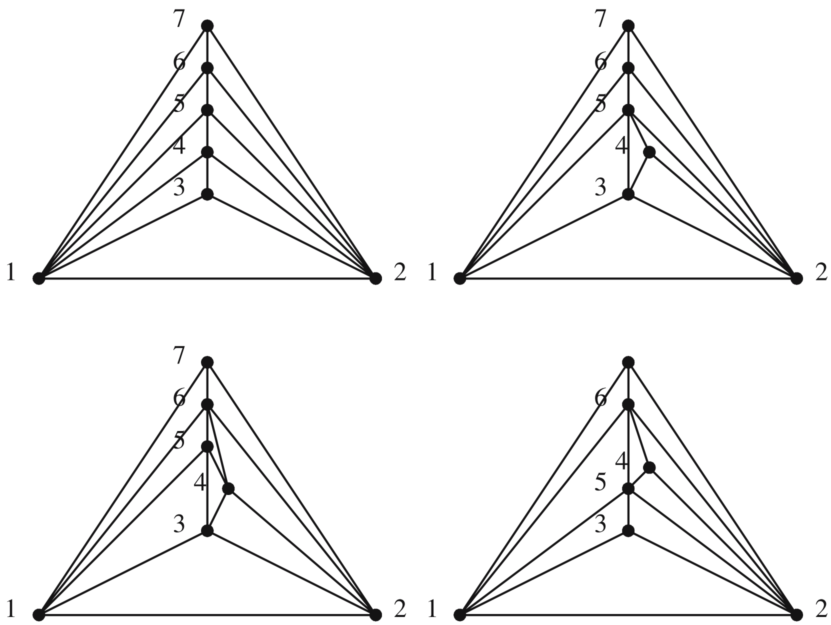

Fig. 4. The four stages of the cycle of 4 flips which regenerate the christmas tree with $\geqslant 7$ nodes, but which show the *absence* of detailed balance. (Top left → top right → bottom left → bottom right → top left.)

An easy computation leads to

$$\prod_{j=1}^{4} \frac{\mathbf{P}(T_j | T_{j+1})}{\mathbf{P}(T_{j+1} | T_j)} = \frac{10}{9},$$

if the number of nodes is larger than six. ∎

4. NUMERICAL SIMULATION

We have performed extensive simulations on the model described above. In this section we summarize the numerical findings, but the reader should note that we have no theoretical explanation for the results. The main insight is that the model with the *uniform* measure[2] leads to an exponential degree distribution, while the model of ref. 4 leads to a power law distribution in a sense which we make clear now, (see Fig. 5).

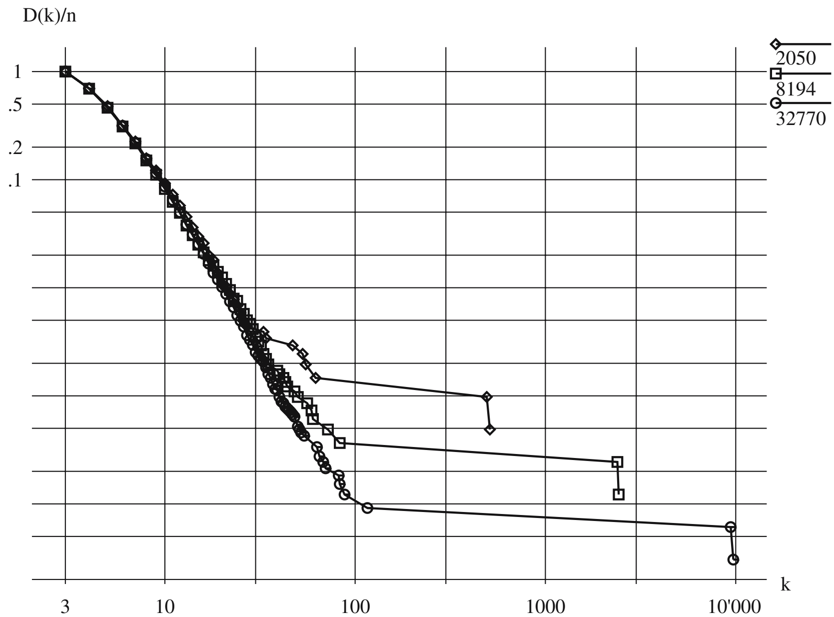

Fig. 5. A log–log plot for triangulations of size $n = 2050$, 8194, and 32770, after about 10^{11} flips. The data are the cumulated sum $D(k)$ of the number of nodes with degree $\geqslant k$, divided by n. The straight part is well fitted with a law of $D(k) \sim ck^{-3}$, so that the degree distribution seems to be $\sim k^{-4}$. We see that there are always two outliers outside the power law on surfaces of genus 0. (This number grows to 3–4 when a new node takes over the leading degree.) We have not been able to reach convergence for triangulations with higher numbers of nodes.

We formulate the results as

Conjecture 4.1. There is a probability measure p on the integers larger than 2 such that the number of nodes of degree d divided by n converges when n tends to infinity to $p(d)$. Moreover p has polynomial decay in the sense that $d^{-4} p(d)$ converges to a non-zero finite limit when d tends to infinity.

Remark 4.2. It should be noted that several deviations from a pure power law are present in these experiments and will not go away with large n. First of all, nodes of degree 3 are less frequent than would be suggested by a power law. We attribute this to the impossibility of doing a flip if a node of degree 3 is chosen: All its edges are unflippable. Second, there is always an outlier with a much higher degree than what is suggested by the power law and most of the time there are two. The reader should note that while Fig. 5 shows exactly two outliers for each triangulation, there are in fact sometimes more, since it is not always the same node which has maximal degree, and thus, several nodes will momentarily "compete" for the highest degree.

We have done extensive checks for the correlations of degrees of neighboring nodes in our simulations. Such correlations have been theoretically explained in ref. 2 for the case of the uniform choosing rule. These correlations are difficult to measure, but no decisive deviation from independence was found, except for some obvious topological rules.

There is a feeling in the community of specialists, be they interested in random triangulations, or in 2-d gravity (the dynamic dual of our problem) that the "typical" triangulation should be "flat" (which means that each node should (wants to?) have six links). To measure the effect of the tails of distribution of degrees, we use combinatorial differential geometry, as advocated by Robin Forman,[1] who introduces a notion of "combinatorial Ricci curvature" which, in our case of triangulations reduces to $\sum_i d_i^2 - 5d_i$. Extensive simulations show that this quantity seems to grow more or less monotonically as the process reaches the equilibrium state. (Note that since $\sum_i d_i$ does not depend on the triangulations, we are just measuring the sum of the squares of the degrees.) Once the "maximum" has been reached, one observes, with high precision, normal fluctuations around this maximum. In fact, we used this as the criterion that convergence has indeed taken place. The maximal degree is about $n/(3.3 \pm 0.2)$ and is reached at two nodes. This means that two of the nodes are each connected to about a third of all nodes. If so, these two outliers will give a contribution proportional to n^2 to the combinatorial Ricci curvature while the rest of the nodes will only contribute linearly if they obey Conjecture 4.1.

Another observation, which holds with very high accuracy is that once a node has been chosen, at equilibrium, exactly 50% of all attempted flips are *not* possible, because the "other" link is already present. This means that a tetrahedron is placed on top of a triangle. Note that the study of such "vertex-insertions" is already present in Tutte's work.[7]

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